On the multiplicity of terminal singularities on threefolds

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Abstract. We give the multiplicity of terminal singularities on threefolds by simple calculation. Then we obtain the best inequalities for the multiplicity and the index. By using this, we can improve the boundedness number of terminal weak $\mathbb{Q}$-Fano 3-folds in [KMMT, Theorem 1.2]. Furthermore, we can extend [K, Theorem 3.6] for Fujita freeness conditions to nonhypersurface terminal singularities.

0 Introduction

Our results are the multiplicity of terminal singularities and the best inequalities for the multiplicity and the index of terminal singularities on threefolds. Our results are partially generalizations of Artin [A]'s result, that, for a normal surface S, a rational singular point p of S, $\mathrm{embdim}_p S = \mathrm{mult}_p S + 1$.

We shall prove the following results in this paper: (Theorem 2.1) Let (X, p) be a 3-fold terminal singular point over $\mathbb{C}$. Then, for all integers k $\dim m^k_{X_p}/m^{k+1}_{X_p} = \mathrm{mult}_p X \cdot k(k+1)/2 + k + 1$ and $\mathrm{embdim}_p X = \mathrm{mult}_p X + 2$ and $\mathrm{mult}_p X \leq \mathrm{index}_p X + 2$ (if $\mathrm{index}_p X = 1$, then $\mathrm{mult}_p X \leq 2$).

We can improve [KMMT Theorem 1.2 (2)] by (Theorem 2.1) to the following: (Theorem 3.4) Let X be a terminal weak $\mathbb{Q}$-Fano 3-fold. Then the following hold. (1) $-K_x \cdot c_2(X) \geq 0$, and hence $I(X)|24!$. (2) Assume further that the anti-canonical morphism $g : X \to \bar{X}$ does not contract any divisors. Then $(-K_X)^3 \leq 6^3 \cdot (2 + 24!)$. (3)The terminal $\mathbb{Q}$-Fano 3-folds are bounded.

We also can extend [K 3.6] by (Theorem 2.1) to the following: (Theorem 4.1) Let X be a normal projective variety of dimension 3, $x_0 \in X$ a nonhypersurface terminal singular point for $\mathrm{index}_{x_0} X = r \geq 2$, and L an ample $\mathbb{Q}$-Cartier divisor such that $K_X + L$ is Cartier at x_0. Assume that there are positive numbers σ_p for $p = 1, 2, 3$ which satisfy the following conditions: (1) $\sqrt[p]{(L)^p \cdot W} \geq \sigma_p$ for any subvariety W of dimension p which contains x_0, (2) $\sigma_1 \geq 1 + 1/r$, $\sigma_2 \geq (1 + 1/r)\sqrt{r+3}$, and $\sigma_3 > (1 + 1/r)\sqrt[3]{r+2}$. Then

$|K_X + L|$ is free at x_0.
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1 Preliminaries

Definition 1.1. Let m_{X_p} be the maximal ideal of p of X. The *embedding dimension* of X at p is the dimension of the Zariski tangent space,

$$\mathrm{embdim}_p X = \dim \frac{m_{X_p}}{m_{X_p}^2}.$$

We basically use the following:

Theorem 1.2 ([A]). *Let S be a normal surface, p be a point of S. Suppose S has a rational singularity at p. Let Z be the fundamental cycle. Then,*

$$\mathrm{mult}_p S = -Z^2, \text{ for all integers } k \ \dim \frac{m_{S_p}^k}{m_{S_p}^{k+1}} = k\,\mathrm{mult}_p S + 1,$$

$$\text{and } \mathrm{embdim}_p S = \mathrm{mult}_p S + 1.$$

We would like to calculate the multiplicity of terminal singularities on threefolds. We shall need the following Mori's classification theorem of terminal singularities in dimension 3.

Theorem 1.3 ([M]). *Let $0 \in X$ be a 3-fold terminal nonhypersurface singular point over $\mathbb{C}$. Then $0 \in X$ is isomorphic to a singularity described by the following list:*
(1)$cA/r, \{xy + f(z, u^r) = 0, f \in \mathbb{C}\{z, u^r\}, (r, a) = 1\} \subset \mathbb{C}^4/\mathbb{Z}_r(a, r-a, r, 1)$,
(2)$cAx/4, \{x^2 + y^2 + f(z, u^2) = 0, f \in \mathbb{C}\{z, u^2\}\} \subset \mathbb{C}^4/\mathbb{Z}_4(1, 3, 2, 1)$,
(3)$cAx/2, \{x^2 + y^2 + f(z, u) = 0, f \in (z, u)^4\mathbb{C}\{z, u\}\} \subset \mathbb{C}^4/\mathbb{Z}_2(1, 2, 1, 1)$,
(4)$cD/2, \{u^2+z^3+xyz+f(x, y) = 0, f \in (x, y)^4\}$, *or* $\{u^2+xyz+z^n+f(x, y) = 0, f \in (x, y)^4, n \geq 4\}, \{u^2 + y^2z + z^n + f(x, y) = 0, f \in (x, y)^4, n \geq 3\} \subset \mathbb{C}^4/\mathbb{Z}_2(1, 1, 2, 1)$,
(5)$cD/3, u^2 + x^3 + y^3 + z^3 = 0$, *or* $\{u^2 + x^3 + yz^2 + f(x, y, z) = 0, f \in (x, y, z)^4\}, \{u^2 + x^3 + y^3 + f(x, y, z) = 0, f \in (x, y, z)^4\} \subset \mathbb{C}^4/\mathbb{Z}_3(1, 2, 2, 3)$,

(6)$cE/2, \{u^2+x^3+g(y,z)x+h(y,z)=0, g,h\in\mathbb{C}\{y,z\}, g,h\in(y,z)^4\}\subset \mathbb{C}^4/\mathbb{Z}_2(2,1,1,1)$.
The equations have to satisfy 2 *obvious conditions:* 1. *The equations define a terminal hypersurface singularity.* 2. *The equations are $\mathbb{Z}_n$-equivariant. (In fact $\mathbb{Z}_n$-invariant, except for $cAx/4$.)*

2 Main Theorem

Theorem 2.1. *Let (X,p) be a 3-fold terminal singular point over $\mathbb{C}$. Then, for all integers k*

$$\dim \frac{{m_{X_p}}^k}{{m_{X_p}}^{k+1}} = \mathrm{mult}_p X\cdot\frac{k(k+1)}{2}+k+1,\ \mathrm{embdim}_p X = \mathrm{mult}_p X+2,$$

$$\textit{and } \mathrm{mult}_p X \le \mathrm{index}_p X + 2\ (\textit{ if } \mathrm{index}_p X = 1,\ \textit{then } \mathrm{mult}_p X\le 2).$$

Moreover, we assume that $(X,p)\cong(xy+f(z,u^r)=0\subset\mathbb{C}^4/\mathbb{Z}_r(a,r-a,r,1),0)$ or $(\mathbb{C}^3/\mathbb{Z}_r(a,r-a,1),0)$ for $(r,a)=1$ and $r>1$. Let $r_i:=\min\{r_{i-1}-a_{i-1},a_{i-1}\}(r_0>r_1>\cdots>r_n=1)$ and $a_i=r_{i-1}(\ mod\, r_i)$ for $r_0=r$, $a_0=a$. Then,

$$\mathrm{mult}_p X = \frac{r_0}{\lfloor r_1\rfloor}+\frac{r_1}{\lfloor r_2\rfloor}+\cdots+\frac{r_{n-1}}{\lfloor\ r_n\ \rfloor}+2\le r+2(=\textit{if and only if } r_1=1).$$

In othercases ((2),(3),(4),(5), or (6) of Theorem 1.3), then $\mathrm{mult}_p X = r+2$.

Proof. Case 0. Let (X,p) be a smooth point. It is clear.
Case 1. Let (X,p) be a Gorenstein terminal singular point. Since we have $t^2+f(x,y,z)=0$, then $(x,y,z)^k/(x,y,z)^{k+1}$ or $t\cdot(x,y,z)^{k-1}/(x,y,z)^k\in m^k/m^{k+1}$. Hence,

$$\dim\frac{m^k_{X_p}}{m^{k+1}_{X_p}} = 2\frac{(k+2)(k+1)}{2}-(k+1) = 2\frac{k(k+1)}{2}+k+1.$$

$$\text{Hence } \mathrm{mult}_p X = 2 \text{ and } \mathrm{embdim}_p X = \mathrm{mult}_p X+2=4.$$

Case 2. Let (X,p) be a terminal quotient singular point of type $\mathbb{C}^3/\mathbb{Z}_r(a,-a,1)$ with $(r,a)=1$ Let $S_i=\mathbb{C}^2/\mathbb{Z}_r(i,1)$ for $i=a,-a$. For $i=a,-a$, we have

$$(xy)^w\cdot\frac{m^{k-w}_{S_{ip}}}{m^{k-w+1}_{S_{ip}}}\in\frac{m^k_{X_p}}{m^{k+1}_{X_p}} \text{ and } \frac{m^{k-w}_{S_{a p}}}{m^{k-w+1}_{S_{a p}}}\cap\frac{m^{k-w}_{S_{-a_p}}}{m^{k-w+1}_{S_{-a_p}}}=(z^r)^{k-w} \text{ for } 0\le w\le k.$$

Then by Theorem 1.2,

$$\dim \frac{m_{X_p}^k}{m_{X_p}^{k+1}} = \sum_{w=0}^{k}\{\dim \frac{m_{S_a p}^{\ k-w}}{m_{S_a p}^{\ k-w+1}} + \dim \frac{m_{S_{-a} p}^{\ k-w}}{m_{S_{-a} p}^{\ k-w+1}} - 1\} =$$

$$\sum_{w=0}^{k}\{(k-w)(\mathrm{mult}_p S_a + \mathrm{mult}_p S_{-a}) + 1\} = (\mathrm{mult}_p S_a + \mathrm{mult}_p S_{-a})\cdot\frac{k(k+1)}{2} + k + 1.$$

Hence, $\mathrm{mult}_p X = \mathrm{mult}_p S_a + \mathrm{mult}_p S_{-a}$ and $\mathrm{embdim}_p X = \mathrm{mult}_p X + 2$.
Since we have

$$\frac{r_i}{r_{i+1}} = (\frac{r_i}{\llcorner r_{i+1} \lrcorner} + 1) - \frac{r_{i+1} - a_{i+1}}{r_{i+1}}, \text{ then}$$

$$\mathrm{mult}_p \mathbb{C}^2/\mathbb{Z}_{r_i}(r_{i+1}, 1) = \frac{r_i}{\llcorner r_{i+1} \lrcorner} - 1 + \mathrm{mult}_p \mathbb{C}^2/\mathbb{Z}_{r_{i+1}}(r_{i+1} - a_{i+1}, 1).$$

Since we have that

$$\frac{r_i}{r_i - r_{i+1}} = 2 - \frac{r_i - 2r_{i+1}}{r_i - r_{i+1}}, \text{ that } \frac{r_i - r_{i+1}}{r_i - 2r_{i+1}} = 2 - \frac{r_i - 3r_{i+1}}{r_i - 2r_{i+1}}, \cdots,$$

$$\text{and that } \frac{r_i - (\llcorner r_i/r_{i+1} \lrcorner - 2)r_{i+1}}{r_i - (\llcorner r_i/r_{i+1} \lrcorner - 1)r_{i+1}} = 2 - \frac{r_i - (\llcorner r_i/r_{i+1} \lrcorner)r_{i+1}}{r_i - (\llcorner r_i/r_{i+1} \lrcorner - 1)r_{i+1}},$$

$$\text{then } \mathrm{mult}_p \mathbb{C}^2/\mathbb{Z}_{r_i}(r_i - r_{i+1}, 1) = 1 + \mathrm{mult}_p \mathbb{C}^2/\mathbb{Z}_{r_{i+1}}(a_{i+1}, 1).$$

Thus $\mathrm{mult}_p \mathbb{C}^2/\mathbb{Z}_{r_i}(r_{i+1}, 1) + \mathrm{mult}_p \mathbb{C}^2/\mathbb{Z}_{r_i}(r_i - r_{i+1}, 1) =$
$\llcorner r_i/r_{i+1} \lrcorner + \mathrm{mult}_p \mathbb{C}^2/\mathbb{Z}_{r_{i+1}}(r_{i+2}, 1) + \mathrm{mult}_p \mathbb{C}^2/\mathbb{Z}_{r_{i+1}}(r_{i+1} - r_{i+2}, 1)$. Hnece,

$$\mathrm{mult}_p X = \frac{r_0}{\llcorner r_1 \lrcorner} + \frac{r_1}{\llcorner r_2 \lrcorner} + \cdots + \frac{r_{n-1}}{\llcorner\ r_n\ \lrcorner} + 2 \leq r + 2 \text{ (last } = \text{ if and only if } r_1 = 1).$$

Case 3. Let (X, p) be a 3-fold terminal nonhypersurface singular point. We shall use the Mori's classification theorem of terminal singularities in dimension 3 ([M]).
Case 3-1. (1)$cA/r, \{xy + f(z, u^r) = 0, f \in \mathbb{C}\{z, u^r\}(r, a) = 1\} \subset \mathbb{C}^4/\mathbb{Z}_r(a, r - a, r, 1)$.
Let $S_i = \mathbb{C}^2/\mathbb{Z}_r(i/r, 1/r)$ for $i = a, -a$. For $i = a, -a$, we have

$$z^w \cdot \frac{m_{S_{i_p}^{k-w}}}{m_{S_{i_p}^{k-w+1}}} \in \frac{m_{X_p}^k}{m_{X_p}^{k+1}} \text{ and } \frac{m_{S_a p}^{\ k-w}}{m_{S_a p}^{\ k-w+1}} \cap \frac{m_{S_{-a} p}^{\ k-w}}{m_{S_{-a} p}^{\ k-w+1}} = (u^r)^{k-w} \text{ for } 0 \leq w \leq k.$$

The rest of the proof is the same as Case 2. Hence, $\mathrm{embdim}_p X = \mathrm{mult}_p X + 2$,

$$\mathrm{mult}_p X = \frac{r_0}{\lfloor r_1 \rfloor} + \frac{r_1}{\lfloor r_2 \rfloor} + \cdots + \frac{r_{n-1}}{\lfloor \ r_n \ \rfloor} + 2 \le \mathrm{index}_p X + 2(\text{ last } = \text{ if and only if } r_1 = 1).$$

Case 3-2. $(2) cAx/4, \{x^2+y^2+f(z,u^2)=0, f \in \mathbb{C}\{z,u^2\}\} \subset \mathbb{C}^4/\mathbb{Z}_4(1,3,2,1)$.
We have $m_{X_p}/m_{X_p}^2 = (yu, yx, u^4, u^2z, z^2, xu^3, xuz, x^2z)$. and $\mathrm{embdim}_p X = 8$.
Then, $(yu)^t u^{4(k-t)-2s} z^s (0 \le s \le 2(k-t)), (yu)^t x u^{4(k-t)-2s-1} z^s (0 \le s \le 2(k-t)-1), (yu)^t (x^2 z)(yx)^s (z^2)^{k-s-t-1} (0 \le s \le k-t-1)$, and $(yu)^t (yx)^s (z^2)^{k-s-t} (1 \le s \le k-t) \in m_{X_p}^k / m_{X_p}^{k+1}$.

$$\dim \frac{m_{X_p}^k}{m_{X_p}^{k+1}} = \sum_{t=0}^{k} \{(2k-2t+1) + (2k-2t) + (k-t) + (k-t)\}$$

$$= \sum_{t=0}^{k} (6k-6t+1) = 6\frac{k(k+1)}{2} + k + 1.$$

Hence, $\mathrm{mult}_p X = \mathrm{index}_p X + 2 = 4 + 2 = 6$ and $\mathrm{embdim}_p X = \mathrm{mult}_p X + 2 = 8$.
Case 3-3. (3)(4)(6)
$(3) cAx/2, \{x^2+y^2+f(z,u)=0, f \in (z,u)^4 \mathbb{C}\{z,u\}\} \subset \mathbb{C}^4/\mathbb{Z}_2(1,2,1,1)$.
We have $m_{X_p}/m_{X_p}^2 = (y, z^2, zu, u^2, xz, xu)$ and $\mathrm{embdim}_p X = 6$.
Then, $y^s (z,u)^{2(k-s)} (0 \le s \le k), x y^t (z,u)^{2(k-t)-1} (0 \le t \le k-1) \in m_{X_p}^k / m_{X_p}^{k+1}$.

$$\dim \frac{m_{X_p}^k}{m_{X_p}^{k+1}} = \sum_{s=0}^{k} (2k-2s+1) + \sum_{t=0}^{k-1} (2k-2t) = 4\frac{k(k+1)}{2} + k + 1.$$

Hence $\mathrm{mult}_p X = \mathrm{index}_p X + 2 = 4$ and $\mathrm{embdim}_p X = \mathrm{mult}_p X + 2 = 6$.
The proofs of (4) and (6) are the same as the proof of (3).
Case 3-4 $(5) cD/3, u^2+x^3+y^3+z^3=0$, or $\{u^2+x^3+yz^2+f(x,y,z)=0, f \in (x,y,z)^4\}$, $\{u^2+x^3+y^3+f(x,y,z), f \in (x,y,z)^4\} \subset \mathbb{C}^4/\mathbb{Z}_3(1,2,2,3)$.
We have $m_{X_p}/m_{X_p}^2 = (xy, xz, u, y^3, y^2z, yz^2, z^3)$. and $\mathrm{embdim}_p X = 7$.
Then, $u^t (x(y,z))^{k-t} (0 \le t \le k), u^t (x(y,z))^{k-t-1} (y,z)^3 (0 \le t \le k-1)$,
$u^t (x(y,z))^s (y,z)^{6+3(k-2-s-t)} (0 \le s+t \le k-2) \in m_{X_p}^k / m_{X_p}^{k+1}$.

$$\dim \frac{m_{X_p}^k}{m_{X_p}^{k+1}} = \sum_{t=0}^{k} (k-t+1) + \sum_{t=0}^{k-1} (k-t+3) + \sum_{t=0}^{k-2} \sum_{s=0}^{k-t-2} 3$$

$$= \frac{1}{2} \cdot (k+1)(k+2) + \frac{1}{2} \cdot k(k+7) + \frac{3}{2} k(k-1) = 5\frac{k(k+1)}{2} + k + 1.$$

Hence $\mathrm{mult}_p X = \mathrm{index}_p X + 2 = 5$ and $\mathrm{embdim}_p X = \mathrm{mult}_p X + 2 = 7$. $\square$

We give the following concrete example:

Example 2.2. *Let* (X,p) *be a quotient singular point of type* $\mathbb{C}^3/\mathbb{Z}_{13}(5,8,1)$. *Then,* $\mathrm{mult}_p X = {}_{\lceil}(13/5)_{\rceil} + {}_{\lceil}(5/2)_{\rceil} + {}_{\lceil}(2/1)_{\rceil} + 2 = 8$.

Theorem 2.1 is wrong on the following canonical singularity on threefolds.

Example 2.3. *Let* (X,p) *be a quotient singular point of type* $\mathbb{C}^3/\mathbb{Z}_3(1,1,1)$. *We have* $\mathrm{mult}_p X = 9$ *and* $\mathrm{embdim}_p X = 10$ *Then,* $\mathrm{embdim}_p X = 10 < \mathrm{mult}_p X + 2 = 11$ *and* $\mathrm{mult}_p X = 9 > \mathrm{index}_p X + 2 = 5$.

3 Application1

We can improve the boundedness number in [KMMT,Theorem 1.2 (2)] by Theorem 2.1.

Definition 3.1 (KMMT Theorem 1.2). Let X be a normal projective variety and X is called a *terminal* (resp.*klt*) $\mathbb{Q}$-Fano variety, if X has only terminal singularities and $-K_X$ is ample. By replacing 'ample' with 'nef and big', *terminal* (resp.*klt*) *weak* $\mathbb{Q}$*-Fano varieties* are similarly defined. Let $I(X)$ be the smallest positive integer I such that IK_X is Cartier; $I(X)$ is called *the Gorenstein index* of X. We note that if X is a klt $\mathbb{Q}$-Fano variety then $|-mK_X|$ is free for some $m > 0$. The induced birational morphism $X \to \bar{X}$ is said to be the *anti-canonical morphism* of X.

Lemma 3.2 ([KMMT Lemma 4.1]). *Let* X *be an* n*-dimensional projective variety and* x *a closed point with multiplicity* r. *Let* D *be a nef and big* $\mathbb{Q}$*-Cartier divisor on* X *and* l *a covering family of curves containing* x *such that* $D \cdot l \leq d$. *Then* $D^n \leq rd^n$.

The following is our improvement for [KMMT Theorem 5.1].

Theorem 3.3. *Let* X *be a* $\mathbb{Q}$*-factorial terminal* $\mathbb{Q}$*-Fano* 3*-fold with* $\rho(X) = 1$. *Then* $(-K_X)^3 \leq 6^3 \cdot\ (2 + 24!)$.

Proof. (cf. [KMMT Theorem 5.1]) By [MM 86, Thm.5], there is a covering family of rational curves $\{l\}$ such that $-K_X \cdot l \leq 6$. If $\{l\}$ has a fixed point x, then Lemma 3.2, we have $(-K_X)^3 \leq 6^3 \cdot \mathrm{mult}_x X$.
We have $\mathrm{mult}_x X \leq 2 + \mathrm{index}_x X$. By [KMMT Theorem 1.2 (1)], we have $\mathrm{index}_x X \leq 24!$. Hence $(-K_X)^3 \leq 6^3 \cdot (2 + 24!)$ in this case.

If $\{l\}$ has a fixed point x, the proof is the same as the one of [KMM92a, Theorem.].

By [KMMT, Construction-Proposition 4.4 and Claim 5.2], there is a covering family of rational curves $\{l'\}$ with a fixed point x such that $-K_X \cdot \{l'\} \leq 3 \times 6$. Hence by Lemma 3.2, $(-K_X)^3 \leq 6^3 \cdot 3^3$ in this case. □

The following is our improvement for [KMMT Theorem 1.2].

Theorem 3.4. *Let X be a terminal weak $\mathbb{Q}$-Fano 3-fold. Then the following hold.* (1) $-K_x \cdot c_2(X) \geq 0$*, and hence* $I(X)|24!$*.* (2) *Assume further that the anti-canonical morphism $g : X \to \bar{X}$ does not contract any divisors. Then* $(-K_X)^3 \leq 6^3 \cdot (2 + 24!)$*.* (3) *The terminal $\mathbb{Q}$-Fano 3-folds are bounded.*

Proof. The proof is the same as the one of [KMMT Theorem 1.2] except that we can use Theorem 3.3 instead of [KMMT Theorem 5.1]. □

4 Application2

We can extend [K,Theorem 3.6] to nonhypersurface terminal singularities in the following.

Theorem 4.1. *Let X be a normal projective variety of dimension 3, $x_0 \in X$ a nonhypersurface terminal singular point for $\mathrm{index}_{x_0} X = r \geq 2$, and L an ample $\mathbb{Q}$-Cartier divisor such that $K_X + L$ is Cartier at x_0. Assume that there are positive numbers σ_p for $p = 1, 2, 3$ which satisfy the following conditions:*
(1) $\sqrt[p]{(L)^p \cdot W} \geq \sigma_p$ *for any subvariety W of dimension p which contains x_0,*
(2) $\sigma_1 \geq 1 + 1/r$*,* $\sigma_2 \geq (1 + 1/r)\sqrt{r + 3}$*, and* $\sigma_3 > (1 + 1/r)\sqrt[3]{r + 2}$*.*
Then $|K_X + L|$ is free at x_0.

Proof. We have $\mathrm{mult}_{x_0} X \leq r + 2$ and $\mathrm{embdim}_{x_0} X \leq r + 4$. The rest of the proof is the same as the one of [K, Theorem 3.6.]. □

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